EXERCISE

EXERCISE 11C continued

b) Show that, according to this model, the peak times for traffic at this motorway junction are, to the nearest minute, 7:45 am and 4:15 pm.

c) Find the traffic flow N expected at these peak times.

d) Between these two rush hour peaks, the traffic is quieter. At what time between the two peaks, according to the model, does the traffic reach its minimum?

e) Find the traffic flow N expected at this minimum.

f) The total number of cars using the junction during one day can be found using $N_{total} = \int_0^{24} N\,\mathrm{d}t$. Using the substitution $u = \frac{t}{6} - 2$, or otherwise, find N_{total}.

EXERCISE

11.5 Integration By Parts

When integrating a product of terms, such as $x \cos x$, it is not always possible to use a substitution to simplify the problem. In such cases, it is often necessary to integrate by parts. The formula is:

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

This process is the reverse of the product rule for differentiation.

PROOF OF THE INTEGRATION BY PARTS FORMULA

You do not need to learn this proof.

Consider $y = uv$ where u and v are both functions of x.

Then, using the product rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Hence: $\int u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = uv$

Remember that when integrating a sum or difference, each term can be integrated separately.

So: $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = uv$

$$\Rightarrow \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

From the product of two terms:

- Choose one term that will be u. We will differentiate this to obtain $\frac{\mathrm{d}u}{\mathrm{d}x}$.
- Choose the other term to be $\frac{\mathrm{d}v}{\mathrm{d}x}$. We will integrate this to find v.

EXAMPLE 1

Find $\int xe^{2x}\,\mathrm{d}x$.

Let: $u = x,\ \frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$

Differentiate u to obtain $\frac{\mathrm{d}u}{\mathrm{d}x}$: $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

Integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$ to obtain v: $\frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$

Using:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

$$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}(1)\,\mathrm{d}x$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

$$= \frac{1}{4}e^{2x}(2x-1) + c$$

How do we decide which term should be u and which $\frac{\mathrm{d}v}{\mathrm{d}x}$? In general, any term that can be differentiated to give a constant should be chosen as u. This, of course, means any multiple of x.

However, if the integral involves a logarithmic term (e.g. $\ln x$, $\log_2 x$, etc.) then this term should be u.

The following list gives, in order of priority, the term that should be chosen as u, if it appears in the integral. You may find it helpful to remember the mnemonic LATE:

1. **L:** Logarithmic terms
2. **A:** Algebraic terms, i.e. terms involving x, x^2, etc
3. **T:** Trigonometric terms: $\sin x$, $\tan x$, etc.
4. **E:** Exponential terms: $e^x, 2^x$, etc.

The integral of $\ln x$ is a special case that requires integration by parts, although it is not obviously a product.

EXAMPLE 2

Find $\int \ln x\ \mathrm{d}x$.

To make it a product of two terms, let: $\int \ln x\ \mathrm{d}x = \int 1.\ln x\ \mathrm{d}x$

Logarithmic terms have highest priority when deciding on u.

Therefore let: $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

continued...